CURIOUS CONNECTIONS: RATIONAL FUNCTION EXTREMA AND THE GEOMETRIC MEAN

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at the PNW-MAA Section Meeting
June 20, 2008
Carroll College, Helena, Montana

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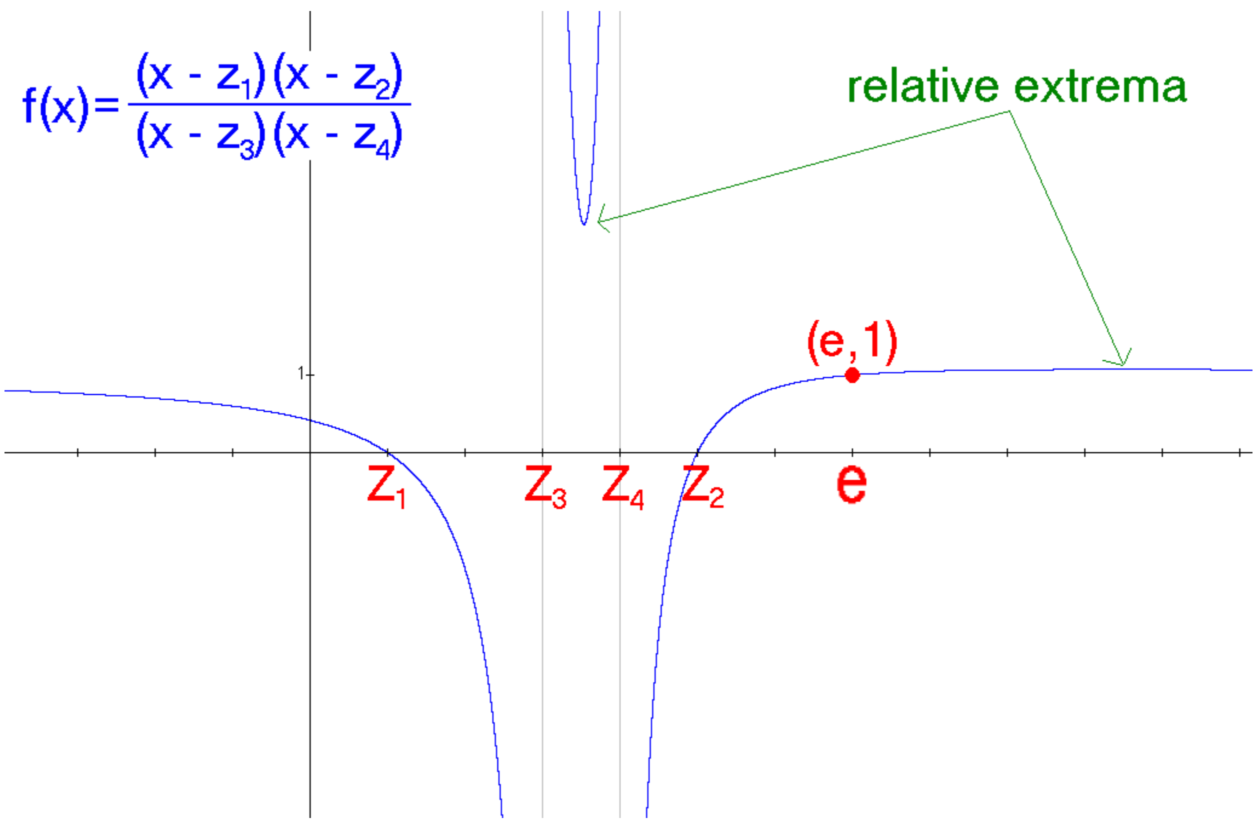
Let $f(x) = \dfrac{x^2+ax+b}{x^2+cx+d} = \dfrac{(x-z_1)(x-z_2)}{(x-z_3)(x-z_4)}$ where the zeroes z_1, z_2, z_3, z_4 may be non-Real.

Let (e,1) be the intersection of the graph of f(x) with its horizontal asymptote, y=1.



Since f(e)=1 we have,

$$f(e)=\frac{e^2+ae+b}{e^2+ce+d}=1$$

$$e^2+ae+b=e^2+ce+d$$

$$ae-ce=d-b$$

$$e(a-c)=d-b$$

$$e=\frac{d-b}{a-c},\ a\neq c. \qquad (1)$$

(we will examine the case of a=c later) and that,

$f(e)=\dfrac{e^2+ae+b}{e^2+ce+d}=\dfrac{(e-z_1)(e-z_2)}{(e-z_3)(e-z_4)}=1$ which gives, $(e-z_1)(e-z_2)=(e-z_3)(e-z_4)$.

THEOREM 1

The relative extrema of f(x) occur at, $x=e\pm\sqrt{(e-z_1)(e-z_2)}=e\pm GM((e-z_1),(e-z_2))$

$$y=\frac{\sqrt{(e-z_1)(e-z_2)}\pm\dfrac{(e-z_1)+(e-z_2)}{2}}{\sqrt{(e-z_3)(e-z_4)}\pm\dfrac{(e-z_3)+(e-z_4)}{2}}=\frac{GM((e-z_1),(e-z_2))\pm AM((e-z_1),(e-z_2))}{GM((e-z_3),(e-z_4))\pm AM((e-z_3),(e-z_4))}$$

where GM and AM refer to the geometric and arithmetic means.

The extrema of f(x) are determined by the five quantities, e, e-z_1, e-z_2, e-z_3, e-z_4, and their geometric and arithmetic means! Too cool!

Let me illustrate with an example, share the proof of the theorem, and conclude with some related results.

EXAMPLE 1. $f(x)=\dfrac{(x-1)(x+1)}{(x-2)(x+3)}=\dfrac{x^2-1}{x^2+x-6}$

The graph of f(x) intersects its horizontal asymptote at e such that f(e) = 1.

So, $e^2-1=e^2+e-6$
$e=5$ This matches (1), $e=\dfrac{d-b}{a-c}=\dfrac{-6+1}{0-1}=5$.

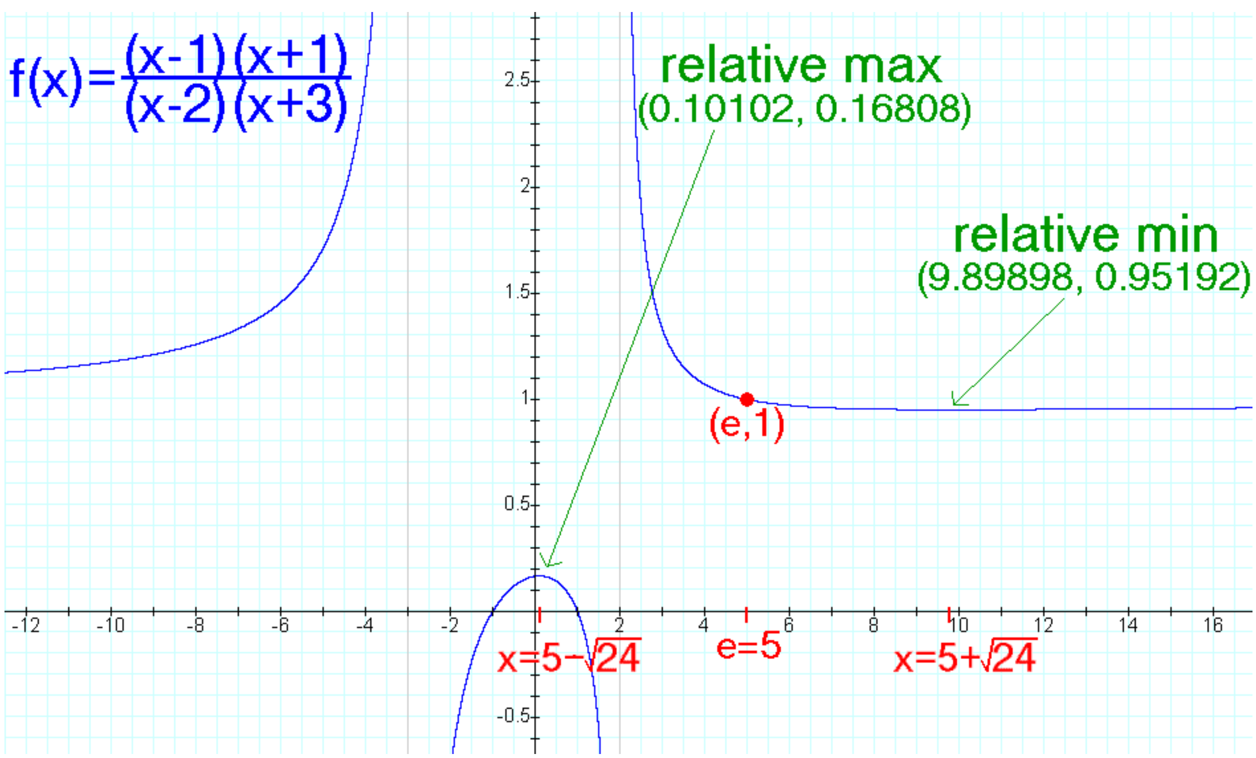
For verification, $f(e)=\dfrac{(e-z_1)(e-z_2)}{(e-z_3)(e-z_4)} \Rightarrow f(5)=\dfrac{(5-1)(5+1)}{(5-2)(5+3)}=\dfrac{4\cdot 6}{3\cdot 8}=1$

Now by theorem 1, we find that the coordinates of the relative extrema are,

$x=e\pm GM((e-z_1),(e-z_2))=5\pm GM(4,6)=5\pm\sqrt{4\cdot 6}=5\pm\sqrt{24}$ and,

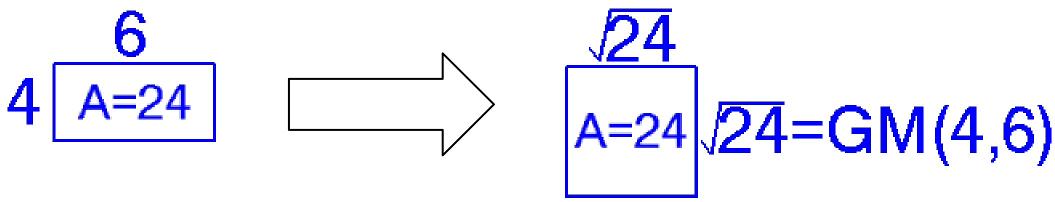
$$y=\frac{GM((e-z_1),(e-z_2))\pm AM((e-z_1),(e-z_2))}{GM((e-z_3),(e-z_4))\pm AM((e-z_3),(e-z_4))}=\frac{GM(4,6)\pm AM(4,6)}{GM(3,8)\pm AM(3,8)}=\frac{\sqrt{4\cdot 6}\pm\frac{4+6}{2}}{\sqrt{3\cdot 8}\pm\frac{3+8}{2}}=\frac{\sqrt{24}\pm 5}{\sqrt{24}\pm\frac{11}{2}}$$

That is, $\left(5+\sqrt{24},\ \dfrac{\sqrt{24}+5}{\sqrt{24}+11/2}\right)$ and $\left(5-\sqrt{24},\ \dfrac{\sqrt{24}-5}{\sqrt{24}-11/2}\right)$. *No calculus required!*

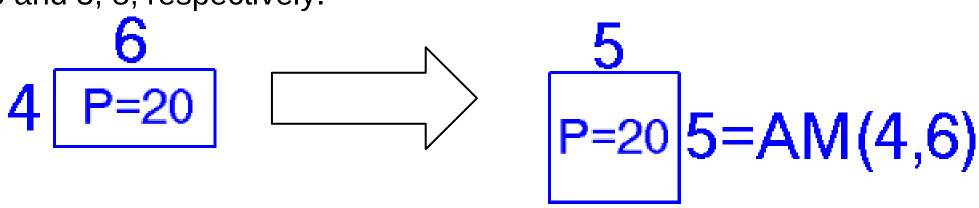


Take a minute to think about the theorem and how it applies to this example. Here, $e-z_1=5-1=4$ and $e-z_2=5-(-1)=6$ are the distances from e to the two x-intercept values, and $e-z_3=5-2=3$ and $e-z_4=5-(-3)=8$ are the distances from e to the two vertical asymptotes.

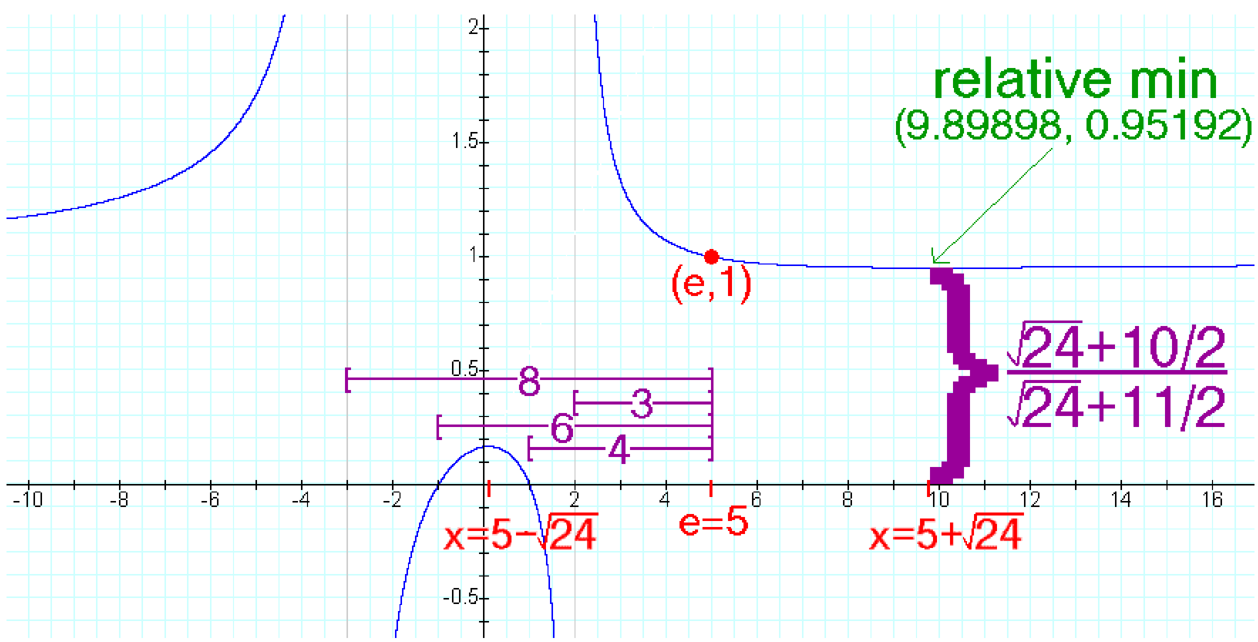
The geometric means of interest are each $\sqrt{24}$. This is the length of a side of the square whose area is equal to the area of the rectangle having side-lengths of the distances above.



The arithmetic means of interest are 10/2 and 11/2. These are the lengths of the sides of the squares whose perimeters equal the perimeters of the rectangles with side-lengths 4, 6 and 3, 8, respectively.



The x-coordinates of the relative extrema are $e \pm GM = 5 \pm \sqrt{24}$. Even more interesting is the y-coordinates of the relative extrema, the ratio of one $GM \pm AM$ over another. Isn't it neat that the height of the function at the extrema points is connected to the squares above whose side-lengths are related to the distances from e to the quadratics' zeros?!



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PROOF OF THEOREM 1.

Let $f(x)=\dfrac{x^2+ax+b}{x^2+cx+d}$, $e=\dfrac{d-b}{a-c}$, $a\neq c$.

Then $f'(x)=\dfrac{(x^2+cx+d)(2x+a)-(x^2+ax+b)(2x+c)}{(x^2+cx+d)^2}$

The only critical values of f(x) occur where $f'(x)=0$ which gives,

$$(x^2+cx+d)(2x+a)-(x^2+ax+b)(2x+c)=0$$
$$(c-a)x^2+2(d-b)x+(ad-bc)=0$$
$$x=\frac{2(b-d)\pm\sqrt{4(d-b)^2-4(c-a)(ad-bc)}}{2(c-a)}$$
$$x=\frac{b-d}{c-a}\pm\sqrt{\frac{(d-b)^2-(c-a)(ad-bc)}{(c-a)^2}}$$

By (1), $e=\dfrac{d-b}{a-c}$, we have,

$$x=e\pm\sqrt{e^2-\frac{ad-bc}{c-a}}$$
$$x=e\pm\sqrt{e^2+\frac{ad-bc}{a-c}}$$
$$x=e\pm\sqrt{e^2+\frac{ad-ab+ab-bc}{a-c}}$$
$$x=e\pm\sqrt{e^2+\left(\frac{d-b}{a-c}\right)a+b}$$
$$x=e\pm\sqrt{e^2+ea+b}$$
$$x=e\pm\sqrt{(e-z_1)(e-z_2)}$$

Thus $x=e\pm GM((e-z_1),(e-z_2))$ are the critical values, x_c, of the graph of f(x) and the extrema of f(x) occur at these values. Further,
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\begin{aligned}
f(x_c) &= \frac{(e \pm \sqrt{(e-z_1)(e-z_2)})^2 + a(e \pm \sqrt{(e-z_1)(e-z_2)}) + b}{(e \pm \sqrt{(e-z_1)(e-z_2)})^2 + c(e \pm \sqrt{(e-z_1)(e-z_2)}) + d} \\
&= \frac{e^2 \pm 2e\sqrt{(e-z_1)(e-z_2)} + (e-z_1)(e-z_2) + ea \pm a\sqrt{(e-z_1)(e-z_2)} + b}{e^2 \pm 2e\sqrt{(e-z_1)(e-z_2)} + (e-z_1)(e-z_2) + ec \pm c\sqrt{(e-z_1)(e-z_2)} + d} \\
&= \frac{e^2 + ea + b + (e-z_1)(e-z_2) \pm \sqrt{(e-z_1)(e-z_2)}(2e+a)}{e^2 + ec + d + (e-z_1)(e-z_2) \pm \sqrt{(e-z_1)(e-z_2)}(2e+c)} \\
&= \frac{(e-z_1)(e-z_2) + (e-z_1)(e-z_2) \pm \sqrt{(e-z_1)(e-z_2)}(2e+a)}{(e-z_3)(e-z_4) + (e-z_3)(e-z_4) \pm \sqrt{(e-z_3)(e-z_4)}(2e+c)} \\
&= \frac{2(e-z_1)(e-z_2) \pm \sqrt{(e-z_1)(e-z_2)}(2e+a)}{2(e-z_3)(e-z_4) \pm \sqrt{(e-z_3)(e-z_4)}(2e+c)} \\
&= \frac{\sqrt{(e-z_1)(e-z_2)} \pm \frac{2e+a}{2}}{\sqrt{(e-z_3)(e-z_4)} \pm \frac{2e+c}{2}} \\
&= \frac{\sqrt{(e-z_1)(e-z_2)} \pm \frac{2e+(-z_1-z_2)}{2}}{\sqrt{(e-z_3)(e-z_4)} \pm \frac{2e+(-z_3-z_4)}{2}} \\
&= \frac{\sqrt{(e-z_1)(e-z_2)} \pm \frac{(e-z_1)+(e-z_2)}{2}}{\sqrt{(e-z_3)(e-z_4)} \pm \frac{(e-z_3)+(e-z_4)}{2}} \\
&= \frac{GM((e-z_1),(e-z_2)) \pm AM((e-z_1),(e-z_2))}{GM((e-z_3),(e-z_4)) \pm AM((e-z_3),(e-z_4))}
\end{aligned}
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Let's look at an example where the graph of f has no vertical asymptotes, but still crosses the horizontal asymptote.

EXAMPLE 2. $f(x)=\frac{x^2+2x-3}{x^2+x+1}=\frac{(x-1)(x+3)}{x^2+x+1}=\frac{(x-1)(x+3)}{\left(x-\frac{-1+i\sqrt{3}}{2}\right)\left(x-\frac{-1-i\sqrt{3}}{2}\right)}$

The graph of f(x) intersects its horizontal asymptote at e such that f(e) = 1.

So, $\begin{aligned} e^2+2e-3&=e^2+e+1 \\ e&=4 \end{aligned}$ This matches (1), $e=\frac{d-b}{a-c}=\frac{1+3}{2-1}=4$.

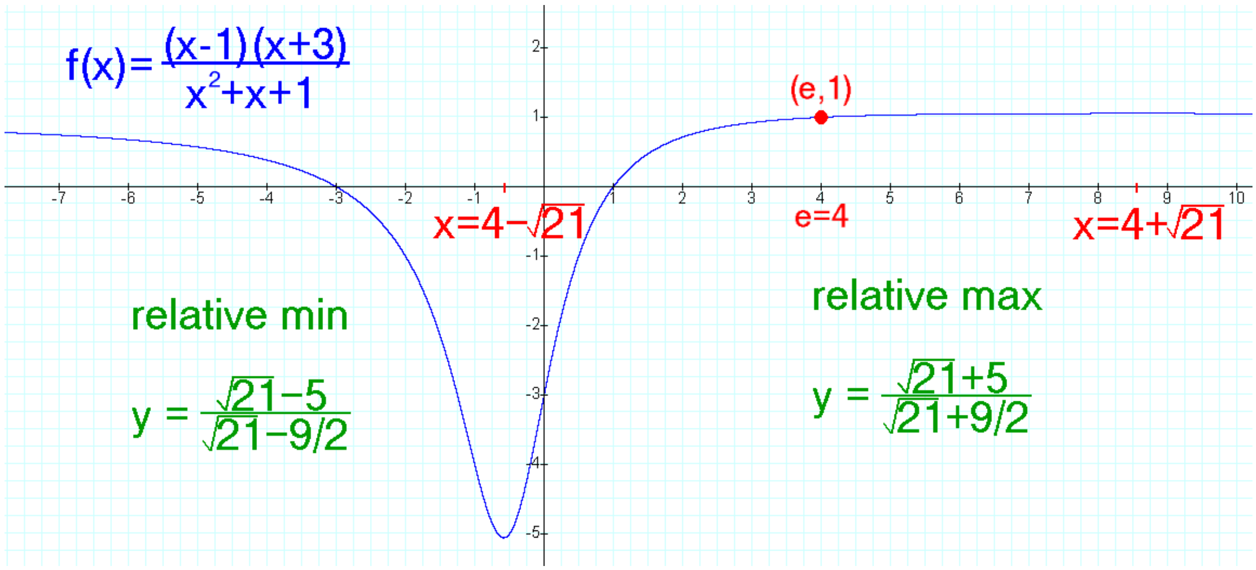
For verification, $f(e)=\frac{(e-z_1)(e-z_2)}{(e-z_3)(e-z_4)} \Rightarrow f(4)=\frac{(4-1)(4+3)}{4^2+4+1}=\frac{3\cdot 7}{21}=1$

Now by theorem 1, we find that the coordinates of the relative extrema are,

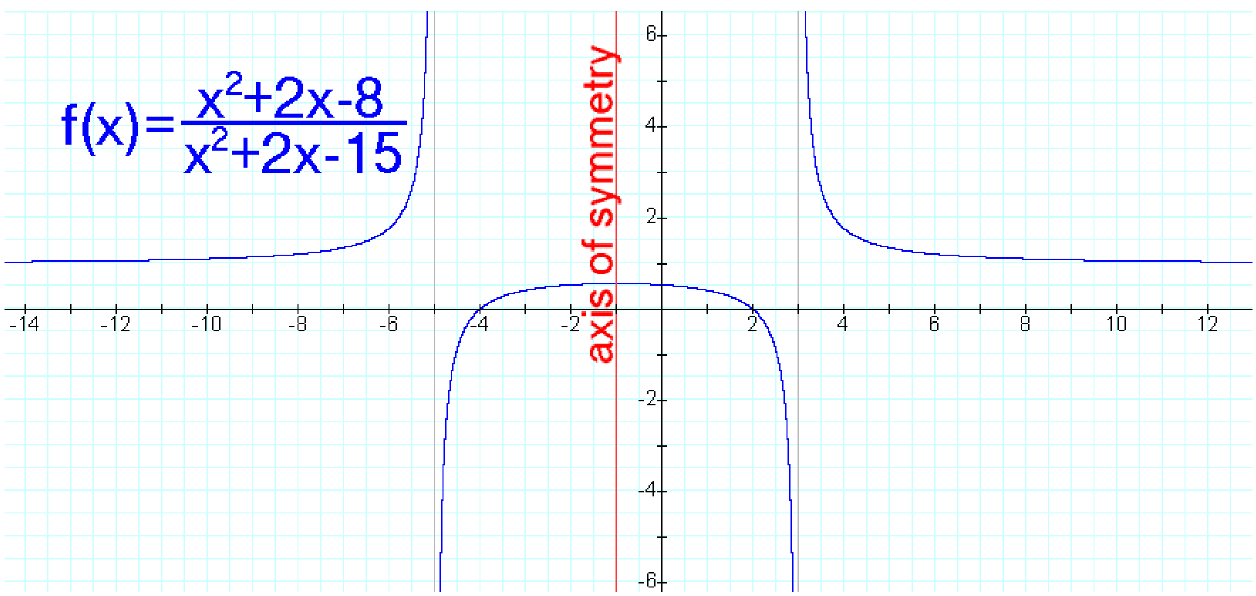
$x=e\pm GM((e-z_1),(e-z_2))=4\pm GM(3,7)=4\pm\sqrt{3\cdot 7}=4\pm\sqrt{21}$ and,

$$\begin{aligned} y&=\frac{GM((e-z_1),(e-z_2))\pm AM((e-z_1),(e-z_2))}{GM((e-z_3),(e-z_4))\pm AM((e-z_3),(e-z_4))} \\ &=\frac{GM(3,7)\pm AM(3,7)}{GM(4-\frac{-1+i\sqrt{3}}{2},4-\frac{-1-i\sqrt{3}}{2})\pm AM(4-\frac{-1+i\sqrt{3}}{2},4-\frac{-1-i\sqrt{3}}{2})} \\ &=\frac{GM(3,7)\pm AM(3,7)}{GM(\frac{9}{2}-\frac{i\sqrt{3}}{2},\frac{9}{2}+\frac{i\sqrt{3}}{2})\pm AM(\frac{9}{2}-\frac{i\sqrt{3}}{2},\frac{9}{2}+\frac{i\sqrt{3}}{2})} \\ &=\frac{\sqrt{3\cdot 7}\pm\frac{3+7}{2}}{\sqrt{\frac{81}{4}-\frac{3i^2}{4}}\pm\frac{\left(9/2-i\sqrt{3}/2\right)+\left(9/2+i\sqrt{3}/2\right)}{2}} \\ &=\frac{\sqrt{21}\pm 5}{\sqrt{21}\pm\frac{9}{2}} \end{aligned}$$

That is, $\left(4+\sqrt{21},\ \frac{\sqrt{21}+5}{\sqrt{21}+9/2}\right)$ and $\left(4-\sqrt{21},\ \frac{\sqrt{21}-5}{\sqrt{21}-9/2}\right)$. *Again, no calcululus needed!*



One last comment. If a=c, the graph of f(x) does not intersect the horizontal asymptote. In this case the graph is symmetric about the line x=-a/2, the midpoint of the two zeroes and also of the two vertical asymptotes (if real) or the real part of the complex zeroes. An example is shown below. The proof is easy, show f(-a/2 – x) = f(-a/2 + x), which we leave to the reader.



Originally written: 2008-02-14
Last revision: 2008-06-16