THE EXTREMA OF THE RATIONAL QUADRATIC FUNCTION $f(x)=(x^2+ax+b)/(x^2+cx+d)$

Larry Larson & Sally Keely

Rational functions and their graphs are explored in most precalculus courses. Many websites provide interactive investigations of the zeroes, vertical asymptotes, and horizontal asymptote of these functions. We present, we think, a surprising relationship between the extrema of the particular rational function $f(x) = \frac{x^2+ax+b}{x^2+cx+d} = \frac{(x-z_1)(x-z_2)}{(x-z_3)(x-z_4)}$ and the zeroes, z_1, z_2, z_3, z_4 of the quadratics involved. We believe the importance of the horizontal asymptote, y=1, has been overlooked in the literature. We provide a method for finding the exact extrema without the use of calculus, one that unexpectedly involves the arithmetic and geometric means of the expressions involving the zeroes.

Let us first present the main theorem then illustrate with an example. The proof of the theorem follows with some additional results. We conclude with some suggestions for further exploration.

THEOREM:

Let $f(x) = \frac{x^2+ax+b}{x^2+cx+d} = \frac{(x-z_1)(x-z_2)}{(x-z_3)(x-z_4)} = \frac{N(x)}{D(x)}$ where the z_i may be real or complex.

Let (e,1) be the point of intersection of f(x) with the horizontal asymptote y=1.

Then $e = \frac{d-b}{a-c}$, $a \neq c$, and the extrema of f(x) occur at (x,y) where

$$x = e \pm \sqrt{(e-z_1)(e-z_2)} = e \pm GM\left((e-z_1),(e-z_2)\right),$$

$$y = \frac{AM\left((e-z_1),(e-z_2)\right) \pm GM\left((e-z_1),(e-z_2)\right)}{AM\left((e-z_3),(e-z_4)\right) \pm GM\left((e-z_3),(e-z_4)\right)},$$

AM refers to the arithmetic mean, and GM refers to the geometric mean.

We illustrate this with an example.

EXAMPLE 1: $f(x) = \frac{(x-1)(x+1)}{(x-2)(x+3)} = \frac{x^2-1}{x^2+x-6}$

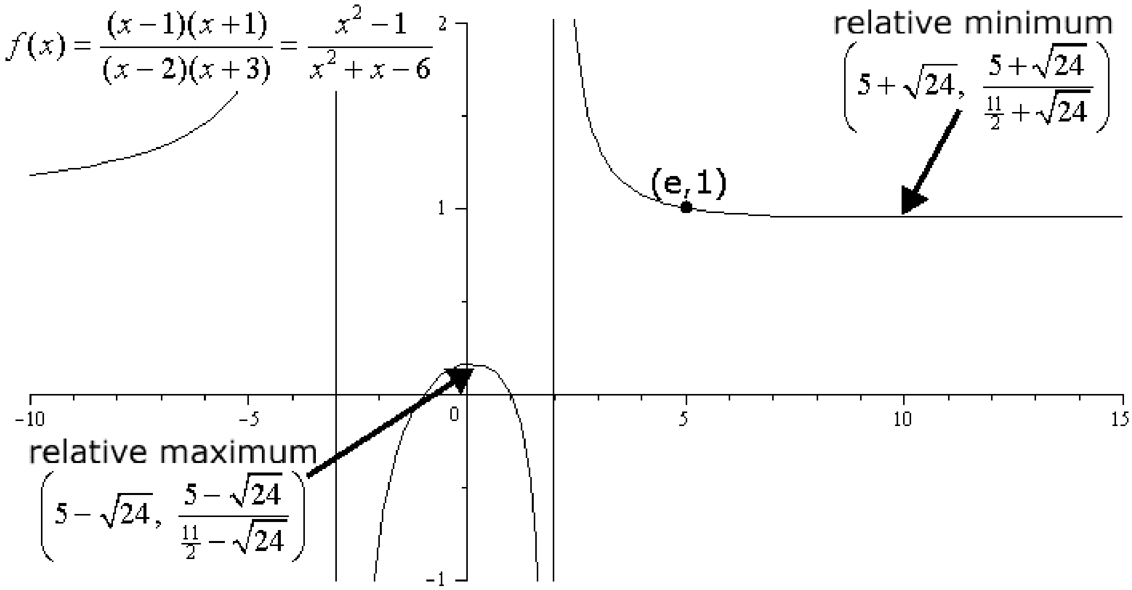
$e = \frac{d-b}{a-c} = \frac{-6+1}{0-1} = 5$ We note that $f(5) = \frac{(4)(6)}{(3)(8)} = 1.$

Now by the theorem, we find that the coordinates of the relative extrema are,

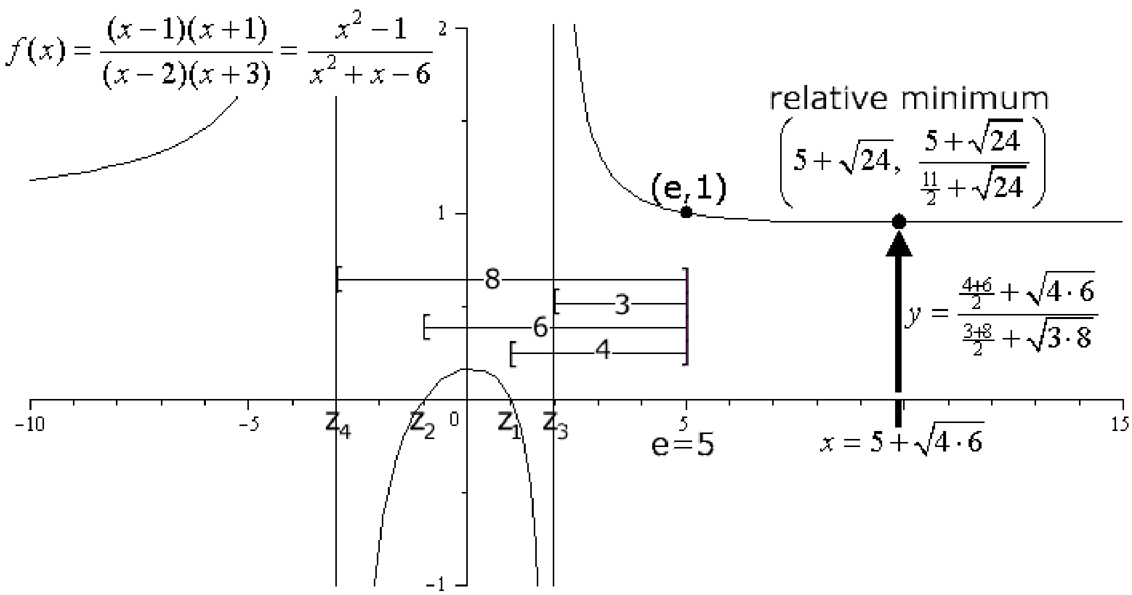
$x = e \pm GM\left((e-z_1),(e-z_2)\right) = 5 \pm GM(4,6) = 5 \pm \sqrt{4 \cdot 6} = 5 \pm \sqrt{24}$ and,

$$y = \frac{AM\left((e-z_1),(e-z_2)\right) \pm GM\left((e-z_1),(e-z_2)\right)}{AM\left((e-z_3),(e-z_4)\right) \pm GM\left((e-z_3),(e-z_4)\right)} = \frac{AM(4,6) \pm GM(4,6)}{AM(3,8) \pm GM(3,8)} = \frac{\frac{4+6}{2} \pm \sqrt{4\cdot 6}}{\frac{3+8}{2} \pm \sqrt{3\cdot 8}} = \frac{5 \pm \sqrt{24}}{\frac{11}{2} \pm \sqrt{24}}$$

That is, $\left(5+\sqrt{24},\ \frac{5+\sqrt{24}}{\frac{11}{2}+\sqrt{24}}\right)$ and $\left(5-\sqrt{24},\ \frac{5-\sqrt{24}}{\frac{11}{2}-\sqrt{24}}\right)$. *No calculus required!*



Note the ease of the calculations using the distances from e to the zeroes.



PROOF OF THEOREM:

Let $f(x) = \frac{x^2+ax+b}{x^2+cx+d} = \frac{(x-z_1)(x-z_2)}{(x-z_3)(x-z_4)} = \frac{N(x)}{D(x)}$

The critical values k of f(x) occur where,

$$f'(k) = 0$$
$$\Rightarrow \; D(k)N'(k) - N(k)D'(k) = 0$$
$$\Rightarrow \; \frac{N(k)}{D(k)} = \frac{N'(k)}{D'(k)}$$
$$\Rightarrow \; \frac{k^2 + ak + b}{k^2 + ck + d} = \frac{2k + a}{2k + c}$$

Cross multiplying and simplifying gives,

$$\Rightarrow \; 2ak^2 + ck^2 + 2bk + ack + bc = 2ck^2 + ak^2 + 2dk + ack + ad$$
$$\Rightarrow \; k^2(a - c) + k(2b - 2d) = ad - bc \qquad \text{(Recall: } e = \tfrac{d-b}{a-c}\text{)}$$

$$\Rightarrow \; k^2 - 2ek + e^2 = \frac{ad - bc}{a - c} + e^2$$
$$\Rightarrow \; (k - e)^2 = \frac{ad - ab + ab - bc}{a - c} + e^2$$
$$\Rightarrow \; (k - e)^2 = \frac{a(d - b)}{a - c} + \frac{b(a - c)}{a - c} + e^2$$
$$\Rightarrow \; (k - e)^2 = ae + b + e^2$$
$$\Rightarrow \; k = e \pm \sqrt{e^2 + ea + b} = e \pm \sqrt{(e - z_1)(e - z_2)}$$

Note if $e^2 + ea + b < 0$ there are no critical values.

We find $f(k) = \dfrac{N(k)}{D(k)} = \dfrac{N'(k)}{D'(k)} = \dfrac{2k + a}{2k + c}$ which implies,

$$f(k) = \frac{k + \frac{a}{2}}{k + \frac{c}{2}}$$
$$= \frac{e \pm \sqrt{(e - z_1)(e - z_2)} + \frac{a}{2}}{e \pm \sqrt{(e - z_3)(e - z_4)} + \frac{c}{2}} \qquad \text{(Note: } e + \tfrac{a}{2} = \tfrac{2e + a}{2} = \tfrac{2e + (-z_1 - z_2)}{2}\text{)}$$
$$= \frac{\dfrac{(e - z_1) + (e - z_2)}{2} \pm \sqrt{(e - z_1)(e - z_2)}}{\dfrac{(e - z_3) + (e - z_4)}{2} \pm \sqrt{(e - z_3)(e - z_4)}}$$
$$= \frac{AM\big((e - z_1),(e - z_2)\big) \pm GM\big((e - z_1),(e - z_2)\big)}{AM\big((e - z_3),(e - z_4)\big) \pm GM\big((e - z_3),(e - z_4)\big)}$$

Let's look at an example where the graph of f has no vertical asymptotes, but still crosses the horizontal asymptote.

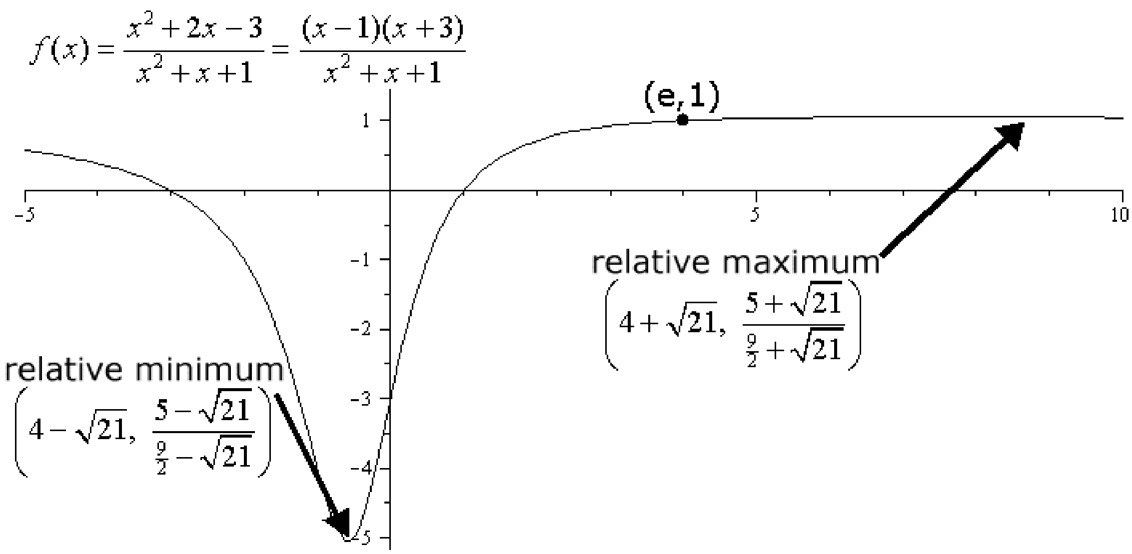
EXAMPLE 2: $f(x) = \dfrac{x^2 + 2x - 3}{x^2 + x + 1} = \dfrac{(x-1)(x+3)}{x^2 + x + 1} = \dfrac{(x-1)(x+3)}{\left(x - \dfrac{-1 + i\sqrt{3}}{2}\right)\left(x - \dfrac{-1 - i\sqrt{3}}{2}\right)}$

$e = \dfrac{d-b}{a-c} = \dfrac{1+3}{2-1} = 4$. Now by the theorem, we find that the coordinates of the relative extrema are,

$x = e \pm GM\left((e - z_1),(e - z_2)\right) = 4 \pm GM\left(3,7\right) = 4 \pm \sqrt{3 \cdot 7} = 4 \pm \sqrt{21}$ and,

$$y = \frac{AM\left((e-z_1),(e-z_2)\right) \pm GM\left((e-z_1),(e-z_2)\right)}{AM\left((e-z_3),(e-z_4)\right) \pm GM\left((e-z_3),(e-z_4)\right)} = \frac{5 \pm \sqrt{21}}{\frac{9}{2} \pm \sqrt{21}}$$

That is, $\left(4 + \sqrt{21},\ \dfrac{5 + \sqrt{21}}{\frac{9}{2} + \sqrt{21}}\right)$ and $\left(4 - \sqrt{21},\ \dfrac{5 - \sqrt{21}}{\frac{9}{2} - \sqrt{21}}\right)$. *Again, no calculus needed.*



Case where a=c:

If a=c then $e = \dfrac{d-b}{a-c} = \pm\infty$. The critical value of f(x), k, is then $\lim_{e \to \pm\infty}\left(e \mp \sqrt{e^2 + ea + b}\right) = \frac{-a}{2} = \frac{-c}{2}$.

PROOF: $\lim_{e \to +\infty}\left(e - \sqrt{e^2 + ea + b}\right) = \lim_{e \to +\infty}\left(\dfrac{-ea - b}{e + \sqrt{e^2 + ea + b}}\right) = \lim_{e \to +\infty}\left(\dfrac{-a - \frac{b}{e}}{1 + \sqrt{1 + \frac{a}{e} + \frac{b}{e^2}}}\right) = \dfrac{-a}{2}$

(The case of the limit as $e \to -\infty$ is left to the reader.)

Now,

$$f\left(\tfrac{-a}{2}\right) = \frac{\left(\tfrac{-a}{2}\right)^2 + a\left(\tfrac{-a}{2}\right) + b}{\left(\tfrac{-a}{2}\right)^2 + c\left(\tfrac{-a}{2}\right) + d}$$

$$= \frac{\tfrac{-a^2}{4} + b}{\tfrac{-c^2}{4} + d} \quad (*)$$

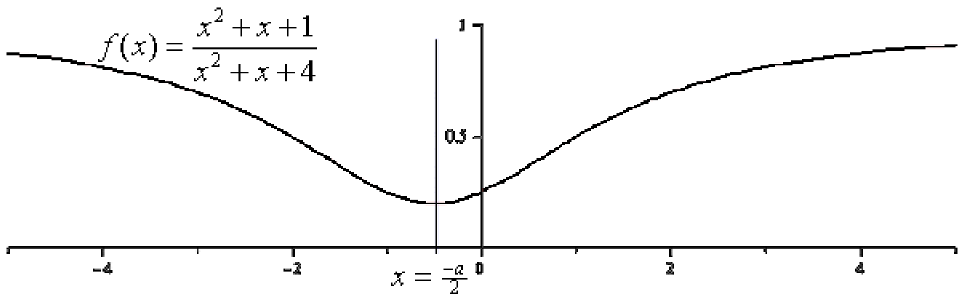
$$= \frac{-\left(\tfrac{-z_1 - z_2}{2}\right)^2 + b}{-\left(\tfrac{-z_3 - z_4}{2}\right)^2 + d}$$

$$= \frac{-\left[AM(z_1, z_2)\right]^2 + \left[GM(z_1, z_2)\right]^2}{-\left[AM(z_3, z_4)\right]^2 + \left[GM(z_3, z_4)\right]^2}$$

$$= \frac{AM^2 - GM^2}{AM^2 - GM^2} \text{ (of the zeros } z_i\text{)}$$

* Interestingly this is $\dfrac{a^2 - 4b}{c^2 - 4d}$, the ratio of the discriminants of N(x) and D(x).

We also note that when a=c, N(x) and D(x) have the same axis of symmetry, and f(x) is symmetric about $x = \frac{-a}{2}$. Showing that $f\left(\frac{-a}{2} + x\right) = f\left(\frac{-a}{2} - x\right)$ we leave as an exercise. We present for observation, the following example:



One further theorem is of interest.

THEOREM: If $a^2 - 4b = c^2 - 4d$, then $f(e + x) = \dfrac{1}{f(e - x)}$.

PROOF: We show that N(e+x) = D(e-x).

$$(e+x)^2 + a(e+x) + b = (e-x)^2 + c(e-x) + d$$
$$4ex + ax + cx + ae - ec + b - d = 0$$
$$x(4e + a + c) + e(a-c) + b - d = 0 \quad \text{(Recall: } e = \tfrac{d-b}{a-c}\text{)}$$
$$x\left(\tfrac{4d-4b+a^2-c^2}{a-c}\right) + d - b + b - d = 0$$
$$x\left(\tfrac{(a^2-4b)-(c^2-4d)}{a-c}\right) = 0$$

Since $a^2 - 4b = c^2 - 4d$ we have $x(0) = 0$

Similarly D(e+x) = N(e-x), thus $f(e+x) = \frac{N(e+x)}{D(e+x)} = \frac{D(e-x)}{N(e-x)} = \frac{1}{f(e-x)}$. In particular, $f_{max} = 1/f_{min}$.

EXAMPLE 3: $f(x) = \dfrac{x^2 - x - 2}{x^2 + 7x + 10} = \dfrac{(x+1)(x-2)}{(x+2)(x+5)}$

$e = \dfrac{d-b}{a-c} = \dfrac{-3}{2}$. $f(e+x) = \dfrac{(\frac{-1}{2}+x)(\frac{-7}{2}+x)}{(\frac{1}{2}+x)(\frac{7}{2}+x)}$ which is the reciprocal of $f(e-x) = \dfrac{(\frac{-1}{2}-x)(\frac{-7}{2}-x)}{(\frac{1}{2}-x)(\frac{7}{2}-x)}$.

By the main theorem, the relative extrema are $\left(\dfrac{-3+\sqrt{7}}{2}, \dfrac{-4+\sqrt{7}}{4+\sqrt{7}}\right)$ .and $\left(\dfrac{-3-\sqrt{7}}{2}, \dfrac{-4-\sqrt{7}}{4-\sqrt{7}}\right)$.

Note that $\dfrac{1}{f_{min}} = \dfrac{1}{\frac{-4+\sqrt{7}}{4+\sqrt{7}}} = \dfrac{4+\sqrt{7}}{-4+\sqrt{7}} = \dfrac{-4-\sqrt{7}}{4-\sqrt{7}} = f_{max}$.

Lastly, we encourage the reader to investigate the following related functions:

1. $f(x) = \dfrac{ax+b}{x^2+cx+d}$ with horizontal asymptote y=0
2. $f(x) = \dfrac{a}{x^2+cx+d}$ with horizontal asymptote y=0
3. $f(x) = \dfrac{ax^2+bx+c}{dx^2+ex+f}$ with horizontal asymptote y=a/d

We have found many interesting geometric nuances related to these functions worthy of further investigation. We wish to thank Russell Gordon of Whitman College, Walla Walla, Washington, for his geometric interpretations which provided us a deeper understanding of our algebraic findings.

Originally written: 2008-02-14
Last revision: 2009-03-29